

Three ways to cover a graph

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Abstract. We consider the problem of covering a host graph G with several graphs from a fixed template class \mathcal{T} . The classical covering number of G with respect to \mathcal{T} is the minimum number of template graphs needed to cover the edges of G . We introduce two new parameters: the local and the folded covering number. Each parameter measures how far G is from the template class in a different way. Whereas the folded covering number has been investigated thoroughly for some template classes, e.g., interval graphs and planar graphs, the local covering number was given only little attention.

We provide new bounds on each covering number w.r.t. the following template classes: linear forests, star forests, caterpillar forests, and interval graphs. The classical graph parameters turning up this way are interval-number, track-number, and linear-, star-, and caterpillar arboricity. As host graphs we consider graphs of bounded degeneracy, bounded degree, or bounded (simple) tree-width, as well as, outerplanar, planar bipartite and planar graphs. For several pairs of a host class and a template class we determine the maximum (local, folded) covering number of a host graph w.r.t. that template class exactly.

1 Introduction

Graph covering is one of the most classical topics in graph theory. In 1891, in one of the first purely graph-theoretical papers at all, Petersen shows that any $2r$ -regular graph can be covered with r sets of vertex disjoint cycles [41]. A first survey on covering problems of Beineke [10] appeared in 1969. Graph covering problems are lively and ramified fields of research – over the last decades as well as today [28,29,2,3,24,39]. This is supported through the course of this paper by many references to recent works of different authors.

In every graph covering problem one is given a host graph G , a template class \mathcal{T} , and a notion of how to cover G with one or several template graphs. One is then interested in covers of G w.r.t. \mathcal{T} that are in some sense simple, or well structured; the most prevalent measure of simplicity being the number of template graphs needed to cover G .

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The main motivation of this paper is to introduce the following three parameters, each of which represents how well G can be covered w.r.t. \mathcal{T} in a different way: The *global covering number*, or simply *covering number*, is the most classical one. All kinds of arboricities, e.g. star [4], caterpillar [23], linear [3], pseudo [42], and ordinary [40] arboricity of a graph are global covering numbers. Other global covering numbers are the (outer) thickness [10,39] and the track-number [26] of a graph. To the best of our knowledge the only *local covering number* in the literature is the bipartite degree introduced by Fishburn and Hammer [20]. Here the *coloring-aspect* is removed from the global covering number, but the underlying covering problem is the same. Finally, the *folded covering number* underlies a different, but related, concept of covering. It has been investigated w.r.t. interval graphs and planar graphs as template class. In the former case the folded covering number is known as the interval-number [30], in the latter case as the splitting-number [32] of a graph.

While some covering numbers, like arboricities, are of mainly theoretical interest, others, like thickness, interval-number and track-number have wide applications in VLSI design [1], network design [43], scheduling and resource allocation [9,12], and bioinformatics [35,33]. The three covering numbers presented here not only unify the notion in the literature, they as well seem interesting in their own right, e.g., provide new approaches to attack or support classical open problems.

In this paper we moreover present new lower and upper bounds for several covering numbers, in particular w.r.t. the template classes: interval graphs, star forests, linear forests, and caterpillar forests, and host classes: graphs of bounded degeneracy or bounded (simple) tree-width, as well as outerplanar, planar bipartite, planar, and regular graphs. We provide an overview over some of our new results in Table 1.

	star forests		caterpillar forests			interval graphs		
	g	$\ell = f$	g	ℓ	f	g	ℓ	f
outer-planar	3 [27]	3	3 [36]	3	3	2 [36]	2	2 [45]
planar + bipartite	4 (T.9)	3 (C.2)	4 [22]	3	3	4	3	3 [45]
planar	5 [5,27]	4 (C.2)	4 [22]	4	4 [45]	4 [23]	?	3 [45]
$\text{stw} \leq k$	$k+1$	$k+1$	$k+1$	$k+1$	$k+1$ (T.8)	$k+1$ (T.7)	k (T.5)	k (T.6)
$\text{tw} \leq k$	$k+1$ [15]	$k+1$	$k+1$	$k+1$	$k+1$	$k+1$	$k+1$	$k+1$ (T.6)
$\text{deg} \leq k$	$2k$ [6,27]	$k+1$ (C.1)	$2k$	$k+1$	$k+1$	$2k$ (T.4)	$k+1$	$k+1$

Table 1. Overview of results. Each row of the table corresponds to a host class \mathcal{G} , each column to a template class \mathcal{T} . Every cell contains the maximum covering number of all graphs $G \in \mathcal{G}$ w.r.t. the template class \mathcal{T} , where the columns labeled g, ℓ, f stand for the global, local and folded covering number, respectively. Grey entries follow by Proposition 1 from other stronger results in the table. Letters T. and C. stand for Theorem and Corollary in the present paper, respectively.

This paper is structured as follows: In order to give a motivating example before the general definition, we start by discussing in Section 2 the linear arboricity and its local and folded variants. In Section 3 the three covering numbers are formally introduced and some general properties are established. In Section 4 we introduce the template classes star forests, caterpillar forests, and interval graphs, and in Section 5 we present our results claimed in Table 1. In Section 6 we briefly discuss the computational complexity of some covering numbers, giving a polynomial time algorithm for the local star arboricity. Moreover, we discuss by how much global, local and folded covering numbers can differ.

2 Folded and Local Linear Arboricity

We give the general definitions of covers and covering numbers in Section 3 below. In this section we motivate and illustrate these concepts on the basis of one fixed template class: the class \mathcal{L} of linear forests, i.e., every graph $L \in \mathcal{L}$ is the disjoint union of simple paths. We want to cover a host graph $G = (V, E)$ by several linear forests $L_1, \dots, L_k \in \mathcal{L}$, i.e., every edge $e \in E$ shall be contained in at least³ one L_i and no non-edge of G shall be contained in any L_i . If G is covered by L_1, \dots, L_k we denote this by $G = \bigcup_{i \in [k]} L_i$.

The *linear arboricity* of G , denoted by $\text{la}(G)$, is the minimum k such that $G = \bigcup_{i \in [k]} L_i$ and $L_i \in \mathcal{L}$ for $i \in [k]$. One easily sees that every graph G of maximum degree Δ has $\text{la}(G) \geq \lceil \frac{\Delta}{2} \rceil$, and every Δ -regular graph G has $\text{la}(G) \geq \lceil \frac{\Delta+1}{2} \rceil$. In 1980, Akiyama et. al. [3] stated the Linear Arboricity Conjecture (LAC). It says that the linear arboricity of any simple graph of maximum degree Δ is either $\lceil \frac{\Delta}{2} \rceil$ or $\lceil \frac{\Delta+1}{2} \rceil$. LAC is confirmed for planar graphs by Wu and Wu [50,51] and asymptotically for general graphs by Alon and Spencer [7]. The general conjecture remains open. The best-known upper bound for $\text{la}(G)$ is $\lceil \frac{3\Delta+2}{5} \rceil$, due to Guldan [25].

We define the *local linear arboricity* of G , denoted by $\text{la}_\ell(G)$, as the minimum j such that $G = \bigcup_{i \in [j]} L_i$ and every vertex v in G is contained in at most j different L_i . Again if G has maximum degree Δ then $\text{la}_\ell(G) \geq \lceil \frac{\Delta}{2} \rceil$, and if G is Δ -regular then $\text{la}_\ell(G) \geq \lceil \frac{\Delta+1}{2} \rceil$. Note that $\text{la}_\ell(G)$ is at most $\text{la}(G)$ and hence the following statement must necessarily hold for LAC to be true.

Conjecture 1. Local Linear Arboricity Conjecture (LLAC): The local linear arboricity of any simple graph with maximum degree Δ is either $\lceil \frac{\Delta}{2} \rceil$ or $\lceil \frac{\Delta+1}{2} \rceil$.

Observation 1 *To prove LAC or LLAC it would suffice to consider Δ -regular graphs with Δ odd: Regularity is obtained by considering a Δ -regular supergraph of G . If Δ is even, say $\Delta = 2k$, one can find a spanning linear forest L_{k+1} in G [25], remove it from the graph, and complement L_{k+1} by a cover L_1, \dots, L_k in the remaining graph on maximum degree $\Delta - 1 = 2k - 1$.*

³ Since linear forests are closed under taking subgraphs, we can indeed assume that $e \in L_i$ for exactly one $i \in [k]$.

If G is Δ -regular with Δ odd, then LLAC states that $G = \bigcup_{i \in [k]} L_i$ with every vertex being an endpoint of exactly one path. While LAC additionally requires that the paths can be colored with $\lceil \frac{\Delta}{2} \rceil$ colors such that no two paths that share a vertex receive the same color. We will see in later sections that sometimes the coloring is the crucial and difficult task.

Next we propose a second way to cover the host graph G with linear forests. A *walk* in G is a not necessarily edge-disjoint path. As before, a set W_1, \dots, W_k of walks covers G , denoted by $G = \bigcup_{i \in [k]} W_i$, if the edge-set E of G is the union of the edge-sets of the walks. We are now interested in how often a vertex v in G appears in the walks W_1, \dots, W_k in total. The *folded linear arboricity* of G , denoted by $\text{la}_f(G)$, is the minimum j such that $G = \bigcup_{i \in [k]} W_i$ and every vertex v in G appears in total at most j times in the walks W_1, \dots, W_k . Again if G has maximum degree Δ then $\text{la}_f(G) \geq \lceil \frac{\Delta}{2} \rceil$, and if G is Δ -regular then $\text{la}_f(G) \geq \lceil \frac{\Delta+1}{2} \rceil$. Clearly, $\text{la}_f(G) \leq \text{la}_\ell(G)$. The next theorem follows directly from a theorem of West [48] (where it is stated in terms of the interval-number $i(G)$). It is a weakening of LLAC above.

Theorem 2. *If G has maximum degree Δ then $\text{la}_f(G) \in \{\lceil \frac{\Delta}{2} \rceil, \lceil \frac{\Delta+1}{2} \rceil\}$.*

Proof. If G is not Eulerian add a vertex x to G and connect it to every vertex in G of odd degree. Consider any Eulerian tour in $G \cup x$ (or G) and (if necessary) split it into shorter walks by removing x from it. \square

Remark 1. Besides LLAC, there are several more weakenings of LAC that are still open. For example⁴, it is open whether the caterpillar arboricity of maximum degree Δ graphs is at most $\lceil \frac{\Delta+1}{2} \rceil$. Yet a weaker, but still open, question asks whether the track-number of these graphs is at most $\lceil \frac{\Delta+1}{2} \rceil$.

3 Covers and Covering Numbers

We formalize the concepts from Section 2 w.r.t. general template classes. For a host graph G and a template class \mathcal{T} , we define a *cover of G w.r.t. \mathcal{T}* to be an edge-surjective homomorphism φ from the disjoint union $T_1 \sqcup T_2 \sqcup \dots \sqcup T_k$ of template graphs, i.e., $T_i \in \mathcal{T}$ for $i \in [k]$, to G . The *size* of a cover is the number of template graphs in the disjoint union. A cover φ is called *injective* if $\varphi|_{T_i}$, that is, φ restricted to T_i , is injective for every $i \in [k]$.

Definition 1. *For a template class \mathcal{T} and a host graph $G = (V, E)$ define the (global) covering number $c_g^\mathcal{T}(G)$, the local covering number $c_\ell^\mathcal{T}(G)$, and the folded covering number $c_f^\mathcal{T}(G)$ as follows:*

$$\begin{aligned} c_g^\mathcal{T}(G) &= \min \{ \text{size of } \varphi : \varphi \text{ injective cover of } G \text{ w.r.t. } \mathcal{T} \} \\ c_\ell^\mathcal{T}(G) &= \min \{ \max_{v \in V} |\varphi^{-1}(v)| : \varphi \text{ injective cover of } G \text{ w.r.t. } \mathcal{T} \} \end{aligned}$$

⁴ See Section 4.2 and 4.3 for the definition of caterpillar arboricity and track-number, respectively.

$$c_f^{\mathcal{T}}(G) = \min \left\{ \max_{v \in V} |\varphi^{-1}(v)| : \varphi \text{ cover of } G \text{ w.r.t. } \mathcal{T} \text{ of size } 1 \right\}$$

Let us rephrase $c_g^{\mathcal{T}}(G)$, $c_\ell^{\mathcal{T}}(G)$, and $c_f^{\mathcal{T}}(G)$: The covering number is the minimum number of template graphs needed to cover the host graph, where covering means identifying subgraphs in G that are template graphs, such that every edge of G is contained in some template graph. In the local covering number the number of template graphs in such a cover is not restricted; Instead the number of template graphs at every vertex should be small. We will see later (and already indicated in Section 2) that these two numbers can differ significantly. The folded covering number is the minimum k such that every vertex v of G can be split into at most k vertices, distributing the incident edges at v arbitrarily (even repeatedly) among them, such that the resulting graph belongs to the template class.

Within the scope of this paper *we only consider template classes that are closed under disjoint union* even without explicitly saying so. E.g., when considering, say stars or cliques, as template graphs we actually mean star forests and collections of cliques, respectively. If the template class \mathcal{T} is closed under disjoint union, then the restriction to covers of size 1 in the definition of $c_f^{\mathcal{T}}$ is unnecessary. This property is also needed to prove the inequality $c_\ell^{\mathcal{T}}(G) \geq c_f^{\mathcal{T}}(G)$ in Proposition 1 (i).

Remark 2. It is still interesting to consider template classes that are *not* closed under disjoint union. For example, if $\mathcal{C}y'$ denotes the class of all simple cycles, then Hajós' Conjecture [37] states that for any n -vertex Eulerian graph G we have $c_g^{\mathcal{C}y'}(G) \leq \lfloor \frac{n}{2} \rfloor$. (It is easily seen that $c_\ell^{\mathcal{C}y'}(G), c_f^{\mathcal{C}y'}(G) \leq \lfloor \frac{n}{2} \rfloor$.)

For any graph parameter one is usually interested in its maximum (or minimum) value within certain graph classes. For $i = g, \ell, f$, a template class \mathcal{T} and a graph class \mathcal{G} , called the *host class*, we define $c_i^{\mathcal{T}}(\mathcal{G}) = \sup \{c_i^{\mathcal{T}}(G) \mid G \in \mathcal{G}\}$. We close this section with a list of inequalities, all of which are elementary applications of Definition 1 and homomorphisms.

Proposition 1. *For template classes $\mathcal{T}, \mathcal{T}'$, host classes $\mathcal{G}, \mathcal{G}'$ and any host graph G we have the following:*

- (i) $c_g^{\mathcal{T}}(G) \geq c_\ell^{\mathcal{T}}(G) \geq c_f^{\mathcal{T}}(G)$
- (ii) *If \mathcal{T} is closed under merging vertices within connected components then $c_\ell^{\mathcal{T}}(G) \leq c_f^{\mathcal{T}}(G)$. In particular with (i) we conclude $c_\ell^{\mathcal{T}}(G) = c_f^{\mathcal{T}}(G)$.*
- (iii) *If $\mathcal{T} \subseteq \mathcal{T}'$ then $c_i^{\mathcal{T}}(G) \geq c_i^{\mathcal{T}'}(G)$ for $i = g, \ell, f$.*
- (iv) *If $\mathcal{G} \subseteq \mathcal{G}'$ then $c_i^{\mathcal{T}}(\mathcal{G}) \leq c_i^{\mathcal{T}}(\mathcal{G}')$ for $i = g, \ell, f$.*
- (v) *If \bar{G} denotes the set of all subgraphs of G and we have $\mathcal{T} \cap \bar{G} \subseteq \mathcal{T}' \cap \bar{G}$, then $c_i^{\mathcal{T}}(G) \geq c_i^{\mathcal{T}'}(G)$ for $i = g, \ell$.*
- (vi) *If \mathcal{T} and \mathcal{T}' are closed under taking subgraphs, then (v) also holds for $i = f$.*

Example In order to exemplify the notions introduced above consider the template class $\mathcal{C}y$ of disjoint unions of cycles. As host graph G we take the Petersen graph.

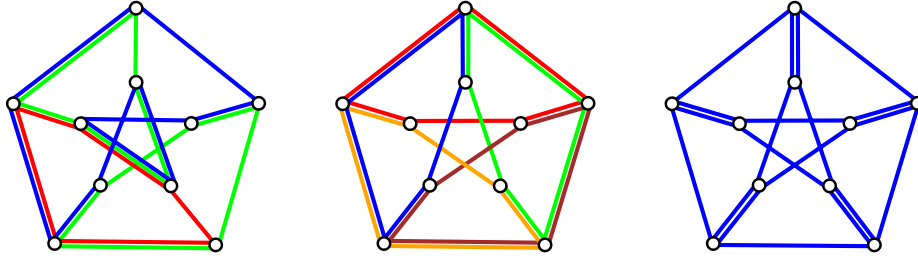


Fig. 1. From left to right: A global cover with three unions of cycles, a local cover of size 5 with three cycles at each vertex, a folded cover with two preimages per vertex. Note that the local cover does not yield a global cover.

Proposition 2. *We have $3 = c_g^{Cy}(G) = c_\ell^{Cy}(G) > c_f^{Cy}(G) = 2$.*

Proof. All witnesses for the upper bounds are shown in Figure 1. Clearly, $c_f^{Cy}(G) \geq 2$ since otherwise G would have to be a disjoint union of cycles. Now suppose, $c_\ell^{Cy}(G) = 2$. Since G is cubic, at each vertex there is exactly one edge contained in two cycles of the covering. Thus, these edges form a perfect matching M of G . Moreover, all cycles involved in the cover are alternating cycles with respect to M . In particular they are all even. Since M is covered twice and the remaining edges of G once, the sum of sizes of cycles in the cover is 20. Hence a 6-cycle C must be involved. Now M restricted to $G \setminus V(C)$ must be perfect. But $G \setminus V(C)$ is a claw. \square

4 Template Classes

In this section we introduce the template classes and covering numbers corresponding to the columns of Table 1. We also include some known results and general observations.

4.1 Star Forests

The star arboricity $sa(G)$ of a graph G , introduced by Akiyama and Kano [4], is the minimum number of star forests, i.e., forests without paths of length 3, into which the edge-set of G can be partitioned. In particular, if S denotes the class of star forests, then $sa(G) = c_g^S(G)$. It is known that outerplanar and planar graphs have star arboricity at most 3 and 5, respectively, see Hakimi et. al. [27]. That this is best possible is shown by Algor and Alon [5]. Moreover, $sa(G) \leq tw(G) + 1$, see Ding et. al. [15] and $sa(G) \leq 2deg(G)$ [27], where $tw(G)$ and $deg(G)$ denote the tree-width (c.f. Section 5.2) and degeneracy (c.f. Section 5.1) of G , respectively. Both bounds are best-possible, see Alon et. al. [6] and Dujmović and Wood [16], respectively.

Since merging any two vertices in a star and omitting double edges yields again a star by Proposition 1 (ii) local and folded star arboricity coincide. In the

following we relate the local star arboricity, denoted by $\text{sa}_\ell(G)$, to two further covering numbers. The *arboricity* $a(G)$, introduced by Nash-Williams [40], is the minimum number of forests needed to cover the edges of G . It is known [40] that $a(G) = \max_{S \subseteq V(G)} \lceil \frac{|E[S]|}{|S|-1} \rceil$. A *pseudoforest* is a graph with at most one cycle per component and the *pseudoarboricity* $p(G)$ is the minimum number of pseudoforests needed to cover the edges of G . Picard and Queyranne [42] notice that $p(G)$ equals the minimum maximum out-degree of all orientations of G , while the latter quantity equals $\max_{S \subseteq V(G)} \lceil \frac{|E[S]|}{|S|} \rceil$, see Frank and Gyárfás [21]. Putting things together one obtains $p(G) \leq a(G) \leq p(G) + 1$. The local star arboricity fits into the picture here.

Theorem 3. *For any graph G we have $p(G) \leq a(G) \leq \text{sa}_\ell(G) \leq p(G) + 1$, where either inequality can be strict. Moreover, $\text{sa}_\ell(G) = p(G)$ if and only if G has an orientation with maximum out-degree $p(G)$ attained only at vertices of degree $p(G)$.*

Proof. Every cover of G w.r.t. stars can be transferred into an orientation of G by orienting every edge towards the center of the corresponding star. If every vertex is contained in at most $\text{sa}_\ell(G)$ stars, then the orientation has maximum out-degree at most $\text{sa}_\ell(G)$, i.e., $p(G) \leq \text{sa}_\ell(G)$.

In the same way every orientation can be transferred into a cover w.r.t. stars by taking at every vertex the star of its incoming edges. If the orientation has maximum out-degree $p(G)$, then each vertex is contained no more than $p(G) + 1$ stars, i.e., $\text{sa}_\ell(G) \leq p(G) + 1$. Moreover, the maximum out-degree is $\text{sa}_\ell(G)$ if and only if for every vertex v that is contained in $\text{sa}_\ell(G)$ stars with centers different from v there is no star with center v . Equivalently, $\text{sa}_\ell(G) = p(G)$ if and only if the maximum out-degree $p(G)$ is attained only at vertices of degree $p(G)$.

To prove $a(G) \leq \text{sa}_\ell(G)$ assume $\text{sa}_\ell(G) = p(G)$. Otherwise (if $\text{sa}_\ell(G) = p(G) + 1$) the result follows from $a(G) \leq p(G) + 1$. Hence, there is an orientation with maximum out-degree $p(G)$ attained only at vertices with degree $p(G)$. Removing these vertices we obtain a graph G' with $p(G') \leq p(G) - 1$, in particular $a(G') \leq p(G)$. We reinsert the vertices of degree $p(G)$ putting each incident edge into a different of the $p(G)$ forests that partition G' . We obtain a cover of G with $p(G)$ forests, i.e., $a(G) \leq p(G) = \text{sa}_\ell(G)$.

Finally, we show that each inequality can be strict: First $k = p(G) < a(G)$ holds for every $2k$ -regular graph. Secondly, we claim that $k = p(G) = \text{sa}_\ell(G)$ holds for the complete bipartite graph $K_{k,n}$ with $n \geq k$. Indeed, $p(K_{k,n}) = \max_{S \subseteq V(K_{k,n})} \lceil \frac{|E[S]|}{|S|} \rceil = \lceil \frac{kn}{k+n} \rceil = k$ and taking all maximal stars with centers in the bigger bipartition class yields $\text{sa}_\ell(K_{k,n}) \leq k$.

It remains to present a graph G with $k = a(G) < \text{sa}_\ell(G)$. It is known [15] that $a(G) \leq \text{tw}(G)$. In Theorem 6 we show⁵ that for every k there is a graph G with $\text{tw}(G) = k$ and $i(G) \geq k + 1$. Then we have $a(G) \leq \text{tw}(G) = k < k + 1 \leq i(G) \leq t_\ell(G) \leq \text{sa}_\ell(G)$, where the next-to-last and last inequality follows by Proposition 1 (i) and (iii), respectively. \square

⁵ $i(G)$ and $t_\ell(G)$ are defined in Section 4.3.

We will derive from Theorem 3 tight upper bounds for the local star arboricity (c.f. Corollary 1 and 2), as well as a polynomial time algorithm to compute the local star arboricity (c.f. Theorem 12).

4.2 Caterpillar Forests

A graph parameter related to the star arboricity is the *caterpillar arboricity* $ca(G)$ of G . A *caterpillar* is a tree in which all non-leaf vertices form a path, called the *spline*. The caterpillar arboricity is the minimum number of caterpillar forests into which the edge-set of G can be partitioned. It has mainly been considered for outerplanar graphs, see Kostochka and West [36], and for planar graphs by Gonçalves and Ochem [22,23]. Since caterpillar forests are exactly triangle-free interval graphs, $ca(G)$ is related to the track-number of G defined below.

4.3 Interval Graphs

The class \mathcal{I} of *interval graphs* has already been considered in many ways and remains present in today's literature. Interval graphs have been generalized to intersection graphs of systems of intervals by several groups of people: Gyárfás and West [26] propose the covering problem w.r.t. the template class \mathcal{I} and introduce the corresponding global covering number called the *track-number*, denoted by $t(G)$, i.e., $t(G) = c_g^{\mathcal{I}}(G)$. It has been shown that outerplanar and planar graphs have track number at most 2 [36] and 4 [23], respectively. Already in 1979 Harary and Trotter [30] introduce the folded covering number w.r.t. interval graphs, called the *interval-number*, denoted by $i(G)$, i.e., $i(G) = c_f^{\mathcal{I}}(G)$. It is known that trees have interval number at most 2 [30], outerplanar and planar graphs have interval number at most 2 and 3, respectively, see Scheinermann and West [45]. All these bounds are tight.

The *local track-number* $t_\ell(G) := c_\ell^{\mathcal{I}}(G)$ is a natural variation of $i(G)$ and $t(G)$, which to our knowledge has not been considered so far.

5 Results

In this section we present all the new results displayed in Table 1. We proceed host class by host class.

5.1 Bounded Degeneracy

The *degeneracy* $\deg(G)$ of a graph G is the minimum of the maximum out-degree over all *acyclic* orientations of G . It is one less than the *coloring number*, introduced in [18], and is a classical measure for the sparsity of G . The next corollary follows directly from the definition of degeneracy and Theorem 3.

Corollary 1. *For every G we have $sa_\ell(G) \leq \deg(G) + 1$.*

Let \mathcal{I} be the class of interval graphs and \mathcal{Ca} be the class of caterpillar forests, i.e., the class of *bipartite* interval graphs. Then by Proposition 1 (v) and (vi) we have $c_i^{\mathcal{I}}(G) = c_i^{\mathcal{Ca}}(G)$ for $i = g, \ell, f$ and every *bipartite* graph G . In particular, if G is bipartite then $t(G) = \text{ca}(G)$ and $i(G) = \text{ca}_f(G)$. In Theorems 4, 6 and 7 below we present graphs with high (folded) caterpillar arboricity. Since all these graphs are bipartite, we obtain lower bounds on the track-number and interval-number of those graphs. Indeed we always define a super-graph G of the complete bipartite graph $K_{m,n}$, for which track- and interval-number has already been determined: $t(K_{m,n}) = \text{ca}(K_{m,n}) = \lceil \frac{mn}{m+n-1} \rceil$ [26] and $i(K_{m,n}) = \text{ca}_f(K_{m,n}) = \lceil \frac{mn+1}{m+n} \rceil$ [30]. After a few definitions we present Lemma 1, which is applied in all our lower bound results.

We refer to the bipartition classes in $K_{m,n}$ of size m and n by A and B , respectively. We need one more definition. For a cover φ of G w.r.t. \mathcal{T} and a subgraph H of G we define the *restriction of φ to H* as the cover ψ of H given by $\psi := \varphi|_{\varphi^{-1}(H)}$. If \mathcal{T} is closed under taking subgraphs, then ψ is also a cover w.r.t. \mathcal{T} .

Lemma 1. *Let G be a graph with an induced $K_{m,n}$, φ be a cover of G w.r.t. \mathcal{Ca} with $s = \max\{|\varphi^{-1}(a)| : a \in A\}$, and ψ be the restriction of φ to the subgraph G' of G after removing all edges in $K_{m,n}$. Then there are at least $n - 2sm$ vertices $b \in B$ such that $|\psi^{-1}(b)| \leq |\varphi^{-1}(b)| - m$.*

Proof. Every vertex in a caterpillar C has at most 2 neighbors on the spline of C . In particular, if $\varphi : C_1 \cup \dots \cup C_k \rightarrow G$ and $a \in A$, then at most $2s$ incident edges at a are covered by spline edges of some C_i . In other words there are at least $n - 2s$ vertices $b \in B$ such that the edge $\{a, b\}$ is covered under φ by a non-spline edge with b being a leaf. Thus, for at least $n - 2sm$ vertices $b \in B$ this is the case w.r.t. to every $a \in A$.

Now if $\{a, b\}$ is covered by some edge e in C_i with b being a leaf, then in the restriction of φ to $G \setminus \{a, b\}$ the number of preimages of b is one less than in φ . This concludes the proof. \square

Theorem 4. *For every $k \geq 1$ there is a bipartite graph G such that $2 \deg(G) \leq 2k \leq \text{ca}(G) = t(G)$.*

Proof. The graph G consists of an induced $K_{k,n}$ with $|A| = k$ and $|B| = n$, where $n > (k-1)\binom{2k-1}{k-1} + 2k(2k-1)$, and for every k -subset S of B an induced $K_{k, (k-1)^2+1}$ with smaller and larger bipartition class S and B_S , respectively. Orienting edges from B to A and from B_S to S for every S proves $\deg(G) \leq k$. Now consider an injective cover $\varphi : C_1 \cup \dots \cup C_s$ of G w.r.t. \mathcal{Ca} and its restriction ψ to the subgraph of G after removing all edges in $K_{k,n}$. Assume for the sake of contradiction that the size s of φ is at most $2k-1$, i.e., $\max\{|\varphi^{-1}(v)| : v \in V(G)\} \leq s \leq 2k-1$. Then by Lemma 1 there is a set $W \subset B$ of at least $n - 2(2k-1)k > (k-1)\binom{2k-1}{k-1}$ vertices, such that $|\psi^{-1}(b)| \leq |\varphi^{-1}(b)| - k \leq s - k \leq k-1$ for every $b \in W$. In particular, every $b \in W$ has a preimage under ψ in at most $k-1$ of the $2k-1$ caterpillar forests. Since $|W| > (k-1)\binom{2k-1}{k-1}$ there

is a k -set S in W whose preimages are contained in at most $k - 1$ caterpillar forests.

In other words, ψ restricted to $G[S \cup B_S]$ is an injective cover of $K_{k, (k-1)^2+1}$ w.r.t. \mathcal{Ca} with size at most $k - 1$, which is impossible as $\text{ca}(K_{k, (k-1)^2+1}) = \lceil \frac{k(k-1)^2+k}{k+(k-1)^2} \rceil = k$, due to [26]. \square

5.2 Bounded (Simple) Tree-width

A k -tree is a graph that can be constructed starting with a $(k + 1)$ -clique and in every step attaching a new vertex to a k -clique of the already constructed graph. We use the term *stacking* for this kind of attaching. The *tree-width* $\text{tw}(G)$ of a graph G is the minimum k such that G is a *partial k -tree*, i.e., G is a subgraph of some k -tree [44].

We consider a variation of tree-width, called *simple tree-width*. A simple k -tree is a k -tree with the extra requirement that there is a construction sequence in which no two vertices are stacked onto the same k -clique. (Equivalently, a k -tree is simple if it has a tree representation of width k in which every $(k - 1)$ -set of subtrees intersects at most 2 tree-vertices.) Now, the *simple tree-width* $\text{stw}(G)$ of G is the minimum k such that G is a partial simple k -tree, i.e., G is a subgraph of some simple k -tree.

For a graph G with $\text{stw}(G) = k$ or $\text{tw}(G) = k$ we fix any (simple) k -tree that is a supergraph of G and denote it by \tilde{G} . Clearly, G inherits a construction sequence from \tilde{G} , where some edges are omitted.

Lemma 2. *For every G we have $\text{tw}(G) \leq \text{stw}(G) \leq \text{tw}(G) + 1$.*

Proof. The first inequality is clear. For the second inequality we show that every k -tree G is a subgraph of a simple $(k + 1)$ -tree H . Whenever in the construction sequence of G several vertices $\{v_1, \dots, v_n\}$ are stacked onto the same k -clique C we consider $C \cup \{v_1\}$ as a $(k + 1)$ -clique in the construction sequence for H . Stacking now v_i onto C can be interpreted as stacking v_i onto $C \cup \{v_{i-1}\}$ and omitting the edge $\{v_{i-1}, v_i\}$. This way we can avoid multiple stackings onto k -cliques by considering $(k + 1)$ -cliques. \square

Simple tree-width endows the notion of tree-width with a more topological flavor. For a graph G we have the following: $\text{stw}(G) \leq 1$ iff G is a linear forest, $\text{stw}(G) \leq 2$ iff G is outerplanar, $\text{stw}(G) \leq 3$ iff G is planar and $\text{tw}(G) \leq 3$ [17].

Simple tree-width also has connections to discrete geometry. In [11] a *stacked polytope* is defined to be a polytope that admits a triangulation whose dual graph is a tree. From that paper one easily deduces that a full-dimensional polytope $P \subset \mathbb{R}^d$ is stacked if and only if $\text{stw}(G(P)) \leq d$. Here $G(P)$ denotes the 1-skeleton of P .

We consider both, graphs with bounded tree-width and graphs with bounded simple tree-width, as host classes since **A)** most of the results for outerplanar graphs are implied by the corresponding result for $\text{stw} \leq 2$, **B)** lower bound results for $\text{stw} \leq 3$ carry over to planar graphs, **C)** our results differ for interval

graphs as template class (c.f. Theorem 5 and 6), and **D**) when the maximum covering numbers are the same for both classes, the lower bounds are slightly stronger when provided by graphs of low *simple* tree-width.

Theorem 5. *For every graph G we have $t_\ell(G) \leq \text{stw}(G)$.*

Proof. If $\text{stw}(G) = 1$, then G is a linear forest and hence an interval graph. If $\text{stw}(G) = 2$, then G is outerplanar and even has track-number at most 2 as shown in [36].

So let $\text{stw}(G) = s \geq 3$. We build an injective cover $\varphi : I_1 \cup \dots \cup I_k \rightarrow G$ with $\varphi^{-1}(v) \leq s$ for every $v \in V(G)$ and $I_i \in \mathcal{I}$ for $i \in [k]$. We use as I_1, \dots, I_k only certain interval graphs, which we call *slugs*: A slug is like a caterpillar with a fixed spine, except that the graph I_i^v induced by the leaves at every spine vertex $v \in I_i$ is a linear forest⁶. The end vertices of a spine are called *spline-ends* and vertices of degree at most 1 in I_i^v are called *leaf-ends*; See Figure 2 for an example. Note that slugs are indeed interval graphs.

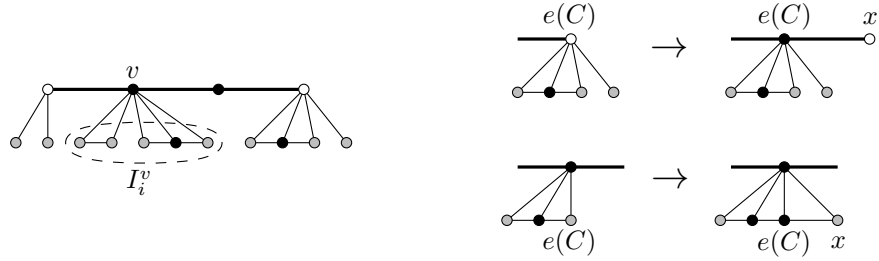


Fig. 2. Left: A slug I_i with the spine drawn thick, spline-ends in white, and leaf-ends in grey. Right: How to extend the slug that contains the end $e(C)$ by a new vertex x when $\{w, w_1\} \in E(G)$ in **Case 1** or $\{w, w_2\} \in E(G)$ in **Case 2**.

We define the cover φ along a construction sequence of G that is inherited from a simple k -tree $\tilde{G} \supseteq G$. At every step let H be the subgraph of G that is already constructed (and hence already covered by φ). We maintain the following invariants on φ , which allow us to stack a new vertex onto every k -clique C onto which no vertex has been stacked so far. We call such a clique *stackable*.

- 1) For every $v \in V(H)$ there is a slug $I(v)$, such that $\varphi^{-1}(v)$ contains a spine vertex of $I(v)$ and $I(v) \neq I(w)$ for $v \neq w$.
- 2) For every stackable clique C there is a spline-end or leaf-end $e(C)$ of some slug I with $\varphi(e(C)) = w_1 \in C$, such that:
 - 2a) If $e(C)$ is a spline-end, then $I \neq I(v)$ for all $v \in V(H)$.
 - 2b) If $e(C)$ is a leaf-end, then $I = I(w_2)$ for some $w_2 \in C \setminus \{w_1\}$.
 - 2c) Every v is $e(C)$ for at most two cliques C , and $v = e(C) = e(C')$ for $C \neq C'$ only if v is a spline-end of degree 0 or a leaf-end of degree 1.

⁶ In a caterpillar I_i^v is an independent set for every spine vertex v .

It is not difficult to satisfy the above invariants for an initial k -clique of \tilde{G} . Indeed, this clique can be build up in a very similar way to the stacking procedure that we describe now: In the construction sequence of G we are about to stack a vertex w onto the stackable clique $C = \{w_1, \dots, w_k\}$ of H , which means that for every $i \in [k]$ the clique $(C \setminus \{w_i\}) \cup \{w\}$ in \tilde{G} is stackable in $H \cup \{w\}$.

For $i = 3, \dots, k$ we do the following. If $\{w, w_i\} \in E(G)$ we introduce a new leaf x to $I(w_i)$ at the corresponding spline vertex in $\varphi^{-1}(w_i)$, and if $\{w, w_i\} \notin E(G)$ we introduce a new slug consisting only of x . Either way, we set $\varphi(x) = w$ and $e((C \setminus \{w_{i-1}\}) \cup \{w\}) = x$. For $i = k$ we additionally set $e((C \setminus \{w_k\}) \cup \{w\}) = x$. For w_1 and w_2 we distinguish two cases, which are illustrated in Figure 2:

In **Case 1** $e(C)$ is a spline-end of some slug I . We first proceed with w_2 as with w_i for $i \geq 3$ above, i.e., $e((C \setminus \{w_1\}) \cup \{w\})$ is set to some new vertex x . Now if $\{w, w_1\} \in E(G)$ we introduce a new spline vertex x to I adjacent to $e(C)$. If $\{w, w_1\} \notin E(G)$ we introduce a new slug I' consisting only of x . We set $\varphi(x) = w$ and $I(w) = I$ if $\{w, w_1\} \in E(G)$ and $I(w) = I'$ otherwise. Note that 1) is satisfied since by 2a) $I \neq I(v)$ for every vertex v in H .

In **Case 2** $e(C)$ is a leaf-end in the slug $I(w_2)$. We introduce a new slug I consisting only of one vertex y and set $\varphi(y) = w$ and $I(w) = I$. Now if $\{w, w_2\} \notin E(G)$ we introduce a new slug I' consisting only of x . If $\{w, w_2\} \in E(G)$ we introduce a new leaf x to $I(w_2)$ at the same spline vertex as $e(C)$. In case $\{w, w_1\} \in E(G)$ the edge $\{y, e(C)\}$ is added. Either way, we set $\varphi(x) = w$ and $e((C \setminus \{w_1\}) \cup \{w\}) = x$.

It is straightforward to check that φ is cover of $H \cup \{w\}$ w.r.t. \mathcal{I} , and that φ satisfies the invariants above. Note that since \tilde{G} is a *simple* k -tree, the clique C is no longer stackable and hence 2) need not be satisfied in $H \cup \{w\}$. Finally, every stackable clique in H different from C was not affected by the above procedure, which completes the proof. \square

We can prove three lower bounds for covering numbers.

Theorem 6. *For every $k \geq 1$ there is a bipartite graph G such that $\text{stw}(G) \leq \text{tw}(G) + 1 \leq k + 1 \leq \text{ca}_f(G) = i(G)$.*

Proof. Our graph G is $K_{k,n}$ with $n = 2k^2 + 1$ where every vertex in the larger bipartition class B has an additional private neighbor. It is easy to see that $\text{tw}(G) \leq k$ and then Lemma 2 yields $\text{stw}(G) \leq \text{tw}(G) + 1$.

Consider any cover φ of G w.r.t. \mathcal{C}_a with $s = \max\{|\varphi^{-1}(v)| : v \in V(G)\}$ and its restriction ψ to the subgraph of G after removing all edges of $K_{k,n}$. By Lemma 1 there are at least $n - 2sk = 2k(k - s) + 1$ vertices $b \in B$ such that $|\psi^{-1}(b)| \leq |\varphi^{-1}(b)| - k$. Since b has a neighbor in $G \setminus K_{k,n}$ we have $|\psi^{-1}(b)| \geq 1$ and hence $s \geq |\varphi^{-1}(b)| \geq k + 1$, i.e., $\text{ca}_f(G) \geq k + 1$. \square

Theorem 7. *For every $k \geq 3$ there is a bipartite graph G such that $\text{stw}(G) + 1 \leq k + 1 \leq \text{ca}(G) = t(G)$.*

Proof. The definition of the graph G starts with a K_{k-1, m_1} with $|B| = m_1 = 2(2k^2 - 2k + 1)$. We denote the vertices in B by $\{u_i, v_i : i \in [\frac{m_1}{2}]\}$. For every

$i \in [\frac{m_1}{2}]$ the graph G contains an induced $K_{2,5(k-1)}$ with smaller and larger bipartition class $A_i = \{u_i, v_i\}$ and $B_i = \{b_1^{ij}, \dots, b_{k-1}^{ij} : j \in [5]\}$. Finally, for every $i \in [\frac{m_1}{2}]$ and $j \in [5]$ there is an induced K_{k-1, m_2} with bipartition classes $A_{ij} = \{b_1^{ij}, \dots, b_{k-1}^{ij}\}$ and B_{ij} with $|B_{ij}| = m_2 = (k-2)^2 + 1$.

Assume for the sake of contradiction that φ is an injective cover of G w.r.t. \mathcal{Ca} of size at most k . Consider the restriction ψ of φ to the subgraph $G' = G \setminus E(K_{k-1, m_1})$ of G . By Lemma 1 there are at least $m_1 - 2k(k-1) = 2k^2 - 2k + 2 > \frac{m_1}{2}$ vertices in $b \in B$ with $|\psi^{-1}(b)| \leq 1$. In particular there is some $i \in [\frac{m_1}{2}]$ such that $|\psi^{-1}(u_i)|, |\psi^{-1}(v_i)| \leq 1$, i.e., in G' each of u_i, v_i is covered by only one caterpillar forest, say $\varphi^{-1}(u_i) \in C_u$ and $\varphi^{-1}(v_i) \in C_v$. Since G' contains a 4-cycle through u_i and v_i we have $C_u \neq C_v$.

Now consider the restriction ϕ of ψ to the subgraph $G'' = G' \setminus E[A_i \cup B_i]$ of G' . Again by Lemma 1 there are at least $5(k-1) - 4$ vertices $b \in B_i$ with $|\phi^{-1}(b)| \leq k-2$; more precisely $C_u, C_v \cap \phi^{-1}(b) = \emptyset$. In particular there is some $j \in [5]$ such that $C_u, C_v \cap \phi^{-1}(b) = \emptyset$ for all $b \in A_{ij}$.

In other words, ϕ restricted to $G[A_{ij} \cup B_{ij}]$ is an injective cover of $K_{k-1, (k-2)^2+1}$ w.r.t. \mathcal{Ca} with size at most $k-2$, which is impossible as $\text{ca}(K_{k-1, (k-2)^2+1}) = \lceil \frac{(k-1)(k-2)^2+k-1}{k-1+(k-2)^2} \rceil = k-1$, due to [26].

It remains to show that $\text{stw}(G) \leq k$. Let $A = \{a_1, \dots, a_{k-1}\}$ be the small bipartition class of K_{k-1, m_1} and $B_{ij} = \{c_1^{ij}, \dots, c_{m_2}^{ij}\}$ for all $i \in [\frac{m_1}{2}], j \in [5]$. Now construct G starting with $A \cup \{u_1, v_1\}$ and the following stackings (where edges not in $E(G)$ are omitted):

- Stack u_i onto $A \cup \{v_{i-1}\}$ and v_i onto $A \cup \{u_i\}$ $\forall i = 2, \dots, \frac{m_1}{2}$
- Stack b_ℓ^{i1} onto $\{a_1, \dots, a_{k-\ell-1}, u_i, v_i, b_1^{i1}, \dots, b_{\ell-1}^{i1}\}$ $\forall i \in [\frac{m_1}{2}], \ell \in [k-1]$
- Stack b_ℓ^{ij} onto $\{u_i, v_i, b_1^{i(j-1)}, \dots, b_{k-\ell-1}^{i(j-1)}, b_1^{ij}, \dots, b_{\ell-1}^{ij}\}$ $\forall i \in [\frac{m_1}{2}], \ell \in [k-1], j \geq 2$
- Stack c_1^{ij} onto $A_{ij} \cup \{u_i\}$ and c_ℓ^{ij} onto $A_{ij} \cup \{c_{\ell-1}^{ij}\}$ $\forall i \in [\frac{m_1}{2}], j \in [5], \ell \geq 2$

Since no k -clique appears twice we conclude that $\text{stw}(G) \leq k$. \square

Theorem 8. *For every $k \geq 2$ there is a graph G such that $\text{stw}(G) + 1 \leq k + 1 \leq \text{ca}_f(G)$.*

Proof. Fix $k \geq 2$. We construct G starting with a star with $k-1$ leaves $\ell_1, \dots, \ell_{k-1}$ and center c_1 , seen as subgraph of a k -clique. For $n := 16k^2 - 16k + 4$ and $i = 2, \dots, n$ stack a new vertex c_i to $\ell_1, \dots, \ell_{k-1}, c_{i-1}$. Now stack vertices s_2, \dots, s_n to $\ell_1, \dots, \ell_{k-2}, c_{i-1}, c_i$. Finally attach a leaf a_i to each of the s_i . This may be viewed as stacking a_i to a subgraph of a k -clique on $\ell_1, \dots, \ell_{k-2}, c_{i-1}, s_i$. By construction $\text{stw}(G) \leq k$.

Assume for the sake of contradiction that $\text{ca}_f(G) \leq k$, i.e., there is a cover φ of G w.r.t. \mathcal{Ca} with $|\varphi^{-1}(v)| \leq k$ for all $v \in V(G)$. We consider three complete bipartite edge-disjoint subgraphs H_1, H_2, H_3 of G induced by:

- $A_1 := \ell_1, \dots, \ell_{k-1}$ and $B_1 := \{c_{2i} \mid 1 \leq i \leq n/2\}$
- $A_2 := \ell_1, \dots, \ell_{k-1}$ and $B_2 := \{c_{2i-1} \mid 1 \leq i \leq n/2\}$

– $A_3 := \ell_1, \dots, \ell_{k-2}$ and $B_3 := \{s_i \mid 2 \leq i \leq n\}$

Note that H_i and H_j are edge-disjoint for $i \neq j$. Denote by ψ the restriction of φ to $G \setminus (E(H_1) \cup E(H_2) \cup E(H_3))$. We apply Lemma 1 three times, once for each H_i , and obtain sets $W_i \subset B_i$ ($i = 1, 2, 3$) with $|W_1|, |W_2| \geq n/2 - 2k(k-1)$ and $|W_3| \geq n - 1 - 2k(k-2)$, and $|\psi_i^{-1}(b)| \leq k - (k-1) = 1$ for $b \in W_i$ ($i = 1, 2$) and $|\psi_3^{-1}(b)| \leq k - (k-2) = 2$ for $b \in W_3$. From the choice of n follows that there are consecutive $c_i, c_{i+1}, c_{i+2}, c_{i+3} \in W_1 \cup W_2$ such that $s_{i+1}, s_{i+2}, s_{i+3} \in W_3$. Together with the leaves $a_{i+1}, a_{i+2}, a_{i+3}$ these vertices induce a 10-vertex graph G' . It is not difficult to check that there is no cover ψ of G' w.r.t $\mathcal{C}a$ with $|\psi^{-1}(c_{i+j})| \leq 1$ for $j = 0, 1, 2, 3$ and $|\psi^{-1}(s_{i+j})| \leq 2$ for $j = 1, 2, 3$ – a contradiction. \square

5.3 Planar and Outerplanar Graphs

Determining maximum covering numbers of (bipartite) planar graphs and outerplanar graphs enjoys a certain popularity as demonstrated by the variety of citations in Table 1. We add three new results to the list.

Theorem 9. *The star arboricity of bipartite planar graphs is at most 4.*

Proof. Every bipartite planar graph can be represented as the contact graph of vertical and horizontal line segments in the plane [13]. Color the edges with four colors depending on whether they are realized by an upper, lower, left, or right end-point of a segment. It is easy to see, that this yields a partition of the edges into four star forests. \square

Corollary 2. *The local star arboricity of planar graphs and bipartite planar graphs is at most 4 and 3, respectively.*

Proof. As mentioned in Section 4.1 the arboricity $a(G)$ of every graph G equals $\max_{S \subseteq V(G)} \lceil \frac{|E[S]|}{|S|} \rceil$ [40]. From this easily follows that every planar graph and planar bipartite graph has arboricity at most 3 and 2, respectively. Now the statement follows from Theorem 3 and $p(G) \leq a(G)$. \square

The only question mark in Table 1 concerns the local track-number of planar graphs. Scheinerman and West [45] show that the interval-number of planar graphs is at most 3, but this is verified with a cover that is not injective. On the other hand, there are bipartite planar graphs with track-number 4 [23]. However by Theorem 9 and Theorem 5 every bipartite planar graph and every planar graph of treewidth at most 3 has local track-number at most 3.

Conjecture 2. The local track-number of a planar graph is at most 3.

6 Complexity and Separability

In Table 1 we provide several pairs of a host class \mathcal{G} and a template class \mathcal{T} for which the global covering number and the local covering number differ, i.e., $c_g^{\mathcal{T}}(\mathcal{G}) > c_\ell^{\mathcal{T}}(\mathcal{G})$. Indeed this difference can be arbitrarily large.

Theorem 10. *For the template class \mathcal{C}_l of collections of cliques and the host class \mathcal{G} of line graphs, we have $c_g^{\mathcal{C}_l}(\mathcal{G}) = \infty$ and $c_\ell^{\mathcal{C}_l}(\mathcal{G}) \leq 2$.*

Proof. By a result of Whitney [49] a graph G is a line graph iff $c_\ell^{\mathcal{C}_l}(G) \leq 2$. To prove $c_g^{\mathcal{C}_l}(\mathcal{G}) = \infty$, we claim that $c_g^{\mathcal{C}_l}(L(K_n)) \in \Omega(\log n)$, i.e., the covering number of the line graph of the complete graph on n vertices is unbounded as n goes to infinity. Assume $L(K_n)$ is covered by k collections of cliques C_1, \dots, C_k . Every clique in $L(K_n)$ corresponds in K_n to either a triangle or a star. We disregard at most $\frac{1}{3}n$ vertices of K_n such that in the induced cover of the smaller line graph no clique in C_1 corresponds to a triangle. Repeating this for every C_i , we end up with a clique cover of $L(K_m)$ with $m \geq (\frac{2}{3})^k n$ that corresponds to a cover of K_m with star forests. Since the star arboricity of K_m is at least $\frac{m-1}{2}$, we get $k \geq \frac{m-1}{2} > (\frac{2}{3})^{k-1} n$, and thus $k \in \Omega(\log n)$. \square

Remark 3. Milans [38] proved a similar result with interval graphs as template class, i.e., $t(L(K_n)) \in \Omega(\log^*(n))$, while $i(G) \leq 2$ for every line graph G .

Table 1 suggests that the separation of the local and the folded covering number is more difficult. Indeed we have $c_\ell^{\mathcal{T}}(\mathcal{G}) = c_f^{\mathcal{T}}(\mathcal{G})$ for every \mathcal{T} and \mathcal{G} in Table 1, except for the local track-number of planar graphs (c.f. Conjecture 2). However, proving upper bounds for $c_\ell^{\mathcal{T}}(\mathcal{G})$ can be significantly more elaborate than for $c_f^{\mathcal{T}}(\mathcal{G})$, even if we suspect that both values are equal; see for example Conjecture 1 and Theorem 2.

Observation 11 *For the template class \mathcal{C}_y of collections of cycles and the host class \mathcal{G} of paths, we have $c_\ell^{\mathcal{C}_y}(\mathcal{G}) = \infty$ and $c_f^{\mathcal{C}_y}(\mathcal{G}) \leq 2$.*

Observation 11 may be considered pathological as there is no *injective* cover of a path P w.r.t. a cycle and hence $c_\ell^{\mathcal{C}_y}(P) = \infty$. However, the local and folded covering number may differ also if $c_\ell^{\mathcal{T}}(G) < \infty$. We have provided one example for this when considering coverings of the Petersen graph with disjoint unions of cycles, see Proposition 2. There is another example: It is known that $i(K_{m,n}) = \lceil \frac{mn+1}{m+n} \rceil$ [30] and $t(K_{m,n}) = \lceil \frac{mn}{m+n-1} \rceil$ [26]. The lower bound on $t(K_{m,n})$ presented in [14] indeed gives $t_\ell(K_{m,n}) \geq \lceil \frac{mn}{m+n-1} \rceil$ and hence we have $t_\ell(K_{m,n}) > i(K_{m,n})$ for appropriate numbers m and n , e.g., $n = m^2 - 2m + 2$. With Proposition 1 this translates into $\text{ca}_\ell(K_{m,n}) > \text{ca}_f(K_{m,n})$.

Another interesting aspect concerns the computational complexity of the three covering numbers. Very informally, one might suspect that the computation of $c_f^{\mathcal{T}}(G)$ is easier than of $c_\ell^{\mathcal{T}}(G)$, which in turn is easier than computing $c_g^{\mathcal{T}}(G)$.

For example, if \mathcal{M} is the class of all matchings, then $c_g^{\mathcal{M}}(G) = \chi'(G)$, the edge-chromatic number of G . Hence deciding $c_g^{\mathcal{M}}(G) \leq 3$ is NP-complete even for 3-regular graphs [31], while $c_\ell^{\mathcal{M}}(G)$ equals the maximum degree of G . As a second example consider the star arboricity $\text{sa}(G)$ and the caterpillar arboricity $\text{ca}(G)$. Deciding $\text{sa}(G) \leq k$ [27,23] and deciding $\text{ca}(G) \leq k$ [23,46] is NP-complete for $k = 2, 3$. The complexity for $k \geq 4$ is unknown in both cases. On the other hand from Theorem 3 we can derive the following.

Theorem 12. *The local star arboricity can be computed in polynomial time.*

Proof. In [21] a flow algorithm is used that given a graph G and $\alpha : V(G) \rightarrow \mathbb{N}$ decides if an orientation D of G exists such that the out-degree of v in D is at most $\alpha(v)$. Moreover, they minimize $\max_{v \in V(G)} \alpha(v)$ such that such D exists in polynomial time. We may use this algorithm to find $p(G)$ in polynomial time. Now define $\alpha(v) := p(G)$ whenever v has degree $p(G)$ and $\alpha(v) := p(G) - 1$ otherwise. By Theorem 3 we have $\text{sa}_\ell(G) = p(G)$ if and only if there exists an orientation D of G satisfying the out-degree constraints given by α . \square

Finally, consider interval graphs as the template class. Shmoys and West [47] and Jiang [34] have shown that deciding $i(G) \leq k$ and $t(G) \leq k$ is NP-complete for every $k \geq 2$, respectively. We claim that the reduction of Jiang also holds for the local track-number. It would be interesting to find a template and host class for which the computation of the folded or local covering number is NP-complete while the global covering number can be computed in polynomial time.

7 Further questions

We have presented new ways to cover a graph and given many example template classes. It is interesting to consider the local and folded variant for more graph covering problems from the literature. For example the covering number w.r.t. (outer)planar graphs is known as the *(outer)thickness* [10] and the folded covering number w.r.t. planar graphs is called the *splitting number* [32]. The local covering number in this case seems unexplored. Further interesting template classes include linear forests of bounded length [8], forests of stars and triangles [19], and chordal graphs.

A concept dual to covering is *packing*. For a host graph G and a template class \mathcal{T} , we define a *packing of G w.r.t. \mathcal{T}* to be an edge-injective homomorphism φ from the disjoint union $T_1 \sqcup T_2 \sqcup \dots \sqcup T_k$ of template graphs, i.e., $T_i \in \mathcal{T}$ for $i \in [k]$, to G . The *size* of a packing is the number of template graphs in the disjoint union. A packing φ is called *injective* if $\varphi|_{T_i}$, that is, φ restricted to T_i , is injective for every $i \in [k]$.

Definition 2. *For a template class \mathcal{T} and a host graph $G = (V, E)$ define the (global) packing number $p_g^{\mathcal{T}}(G)$, the local packing number $p_\ell^{\mathcal{T}}(G)$, and the folded packing number $p_f^{\mathcal{T}}(G)$ as follows:*

$$p_g^{\mathcal{T}}(G) = \max \{ \text{size of } \varphi : \varphi \text{ injective packing of } G \text{ w.r.t. } \mathcal{T} \}$$

$$p_\ell^\mathcal{T}(G) = \max \left\{ \min_{v \in V} |\varphi^{-1}(v)| : \varphi \text{ injective packing of } G \text{ w.r.t. } \mathcal{T} \right\}$$

$$p_f^\mathcal{T}(G) = \max \left\{ \min_{v \in V} |\varphi^{-1}(v)| : \varphi \text{ packing of } G \text{ w.r.t. } \mathcal{T} \text{ of size } 1 \right\}$$

Let us rephrase $p_g^\mathcal{T}(G)$, $p_\ell^\mathcal{T}(G)$, and $p_f^\mathcal{T}(G)$: The packing number is the maximum number of template graphs that can be packed into the host graph, where packing means identifying edge-disjoint subgraphs in G that are template graphs. In the local covering number the number of template graphs in such a packing is not measured; instead the number of template graphs at every vertex should be large. The folded packing number is the maximum k such that every vertex v of G can be split into k vertices, distributing the incident edges at v arbitrarily (not repeatedly) among them, such that the resulting graph is in \mathcal{T} . A classical packing problem is given by \mathcal{T} being the class of non (outer-)planar graphs. In this case the global packing number is called (*outer*) *coarseness* [10].

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